

High Order Perturbative QCD Approach to Multiplicity Distributions of Quark and Gluon Jets

I.M. Dremin¹, C.S. Lam², and V.A. Nechitailo¹

¹*Lebedev Physical Institute, Moscow 117924, Russia*

²*McGill University, Montreal H3A 2T8, Canada*

Abstract

The second and third factorial moments of the multiplicity distributions of gluon and quark jets are calculated up to the next-to-next-to-next-to leading order in perturbative QCD, using the equations for generating functions. The results are confronted with experimental data. A general discussion on high order corrections revealed by such an approach is given. Other possible corrections and approaches are discussed as well.

I. INTRODUCTION

The detailed properties of multiplicity distributions of gluon and quark jets are studied in experiment nowadays [1–3]. Theoretical description of the data in the framework of perturbative QCD is rather successful qualitatively, often at the accuracy of 10–15% or better. Such an accuracy, obtained sometimes only in low-order approximations, is somewhat surprising because the QCD expansion parameter is rather large at present energies (about 0.5), so higher order contributions should be estimated. It happens that different physical quantities are sensitive in a different way to higher order corrections. A thorough discussion of average multiplicities of gluon and quark jets is given in [4–8] with respect to their dependence on these corrections. Here we extend these studies to the widths and higher moments of multiplicity distributions of gluon and quark jets, calculating them in the next-

to-next-to-next-to leading order approximation. Such an accuracy becomes possible because the equations for the generating functions admit a perturbative expansion. Besides finding out the appropriate coefficients, we also discuss some general problems, revealed by such an analysis of average multiplicities, widths, and higher moments.

II. WIDTHS OF MULTIPLICITY DISTRIBUTIONS IN PERTURBATIVE QCD

Any moment of the parton multiplicity distribution of gluon and quark jets in QCD can be obtained from the equations for the generating functions

$$G'_G = \int_0^1 dx K_G^G(x) \gamma_0^2 [G_G(y + \ln x) G_G(y + \ln(1-x)) - G_G(y)] \\ + n_f \int_0^1 dx K_G^F(x) \gamma_0^2 [G_F(y + \ln x) G_F(y + \ln(1-x)) - G_G(y)], \quad (1)$$

$$G'_F = \int_0^1 dx K_F^G(x) \gamma_0^2 [G_G(y + \ln x) G_F(y + \ln(1-x)) - G_F(y)], \quad (2)$$

where G_i are the generating functions of the multiplicity distributions $P_n^{(i)}$ of gluon ($i = G$) and quark ($i = F$) jets, defined by

$$G_i(y, z) = \sum_{n=0}^{\infty} (z+1)^n P_n^{(i)} = \sum_{q=0}^{\infty} \frac{z^q}{q!} \langle n_i \rangle^q F_q^{(i)}. \quad (3)$$

In these expressions, $\langle n_i \rangle = \sum_{n=0}^{\infty} n P_n^{(i)}$ is the average multiplicity, z is an auxiliary variable, $y = \ln(p\Theta/Q_0)$ is the evolution variable, p, Θ are the momentum and the opening angle of a jet, $Q_0 = \text{const}$, $G'(y) = dG/dy$, and n_f is the number of active flavors. Moreover,

$$\gamma_0^2(y) = \frac{2N_c \alpha_S(y)}{\pi}, \quad \alpha_S(y) = \frac{2\pi}{\beta_0 y} \left(1 - \frac{\beta_1 \ln 2y}{\beta_0^2 y} \right), \quad (4)$$

$$\beta_0 = \frac{11N_c - 2n_f}{3}, \quad \beta_1 = \frac{17N_c^2 - n_f(5N_c + 3C_F)}{3}, \quad (5)$$

α_S is the running coupling strength, N_c is the number of colours, and $C_F = (N_c^2 - 1)/2N_c = 4/3$ in QCD. The argument of γ_0^2 in the integrals is chosen to be $y + \ln x(1-x)$, as determined by the transverse momentum of partons at the splitting vertex. The kernels of the equations are

$$K_G^G(x) = \frac{1}{x} - (1-x)[2-x(1-x)], \quad (6)$$

$$K_G^F(x) = \frac{1}{4N_c}[x^2 + (1-x)^2], \quad (7)$$

$$K_F^G(x) = \frac{C_F}{N_c} \left[\frac{1}{x} - 1 + \frac{x}{2} \right]. \quad (8)$$

The normalized factorial moment of any rank q can be obtained by differentiation

$$F_q^{(i)} = \frac{1}{\langle n_i \rangle^q} \frac{d^q G_i}{dz^q} \Big|_{z=0}, \quad (9)$$

or, equivalently, by using the series (3) and collecting the terms with equal powers of z on both sides of the equations (1), (2).

The normalized second factorial moment F_2 defines the width of the multiplicity distribution, and is related to its dispersion $D^2 = \langle n^2 \rangle - \langle n \rangle^2$ by the formula

$$D^2 = (F_2 - 1)\langle n \rangle^2 + \langle n \rangle = K_2 \langle n \rangle^2 + \langle n \rangle, \quad (10)$$

where K_2 is the second cumulant.

The second factorial moments normalized to their own average multiplicities squared are

$$F_2^G = \frac{\langle n_G(n_G - 1) \rangle}{\langle n_G \rangle^2}, \quad F_2^F = \frac{\langle n_F(n_F - 1) \rangle}{\langle n_F \rangle^2}. \quad (11)$$

Let us write down their perturbative expansions up to γ_0^3 -terms as

$$F_2^G = \frac{4}{3}(1 - f_1\gamma_0 - f_2\gamma_0^2 - f_3\gamma_0^3), \quad (12)$$

$$F_2^F = (1 + \frac{r_0}{3})(1 - \phi_1\gamma_0 - \phi_2\gamma_0^2 - \phi_3\gamma_0^3), \quad (13)$$

where $r_0 = N_c/C_F$ determines the asymptotical value of the ratio of multiplicities in gluon and quark jets (see (18) below). It is equal to 9/4 in QCD and to 1 in SUSY QCD. Actually, the asymptotical values of F_2^G and F_2^F in front of the brackets in (12), (13) are found out from the equations below by equating the leading terms on both sides. However, we have inserted their explicit expressions directly here to simplify further notations.

Using the Taylor series expansion of G 's as proposed in [9], one can rewrite the Eqs. (1), (2) as

$$\begin{aligned} \frac{1}{\gamma_0^2} [\ln G_G]'' &= G_G - 1 - 2G'_G v_1 + G''_G v_2 + 0.5G'''_G v_3 + \\ &\quad \left(\frac{G'^2_G}{G_G} \right)' v_{12} + 4B\gamma_0^2 [(G_G - 1)v_1 - G'_G v_2] + \\ \frac{n_f}{4N_c} &\left[\left(\frac{G_F^2}{G_G} \right)' v_4 + 2 \left(\frac{G_F G'_F}{G_G} \right)' v_5 + \left(\frac{G_F G''_F}{G_G} \right)' v_6 + \left(\frac{G_F^2}{G_G} \right)' v_{13} - \right. \\ &\quad \left. 2B\gamma_0^2 \left(\left(\frac{G_F^2}{G_G} - 1 \right) v_4 + 2 \left(\frac{G_F G'_F}{G_G} + \left(\frac{G_F^2}{G_G} \right)' \right) v_5 \right) \right], \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{r_0}{\gamma_0^2} [\ln G_F]'' &= G_G - 1 - G'_G v_7 - G''_G v_8 - 0.5G'''_G v_9 - \\ &\quad \left(\frac{G_G G'_F}{G_F} \right)' v_{10} - 0.5 \left(\frac{G_G G''_F}{G_F} \right)' v_{11} - \left(\frac{G'_G G'_F}{G_F} \right)' v_{14} + \\ &\quad 2B\gamma_0^2 [(G_G - 1)v_7 + 2G'_G v_8 + (G'_G + \frac{G_G G'_F}{G_F})v_{10}]. \end{aligned} \quad (15)$$

The terms up to the third derivative of G are kept everywhere because each derivative gives rise to the factors γ or γ_F as seen from their definition below in Eq. (16), and we are interested in corrections up to γ_0^3 . We use $B = \beta_0/8N_c$, $B_1 = \beta_1/4\beta_0 N_c$. The corresponding expansions for the anomalous QCD dimensions γ and γ_F defined as

$$\langle n_G \rangle \propto \exp\left(\int^y \gamma(y') dy'\right), \quad \langle n_F \rangle \propto \exp\left(\int^y \gamma_F(y') dy'\right) \quad (16)$$

are

$$\gamma = \gamma_0(1 - a_1\gamma_0 - a_2\gamma_0^2 - a_3\gamma_0^3) \quad (17)$$

with $\gamma_F = \gamma - r'/r$, where

$$r = \langle n_G \rangle / \langle n_F \rangle = r_0(1 - r_1\gamma_0 - r_2\gamma_0^2 - r_3\gamma_0^3). \quad (18)$$

All the coefficients a_i, r_i have been calculated and tabulated in [7]. The integrals v_i and terms in the right-hand sides of equations (14), (15) proportional to $\langle n_G \rangle^2 z^2/2$ are given in the Appendix. The corresponding terms in the left-hand sides can be written as

$$[\ln G_G]'' = \frac{\gamma_0^2 \langle n_G \rangle^2 z^2}{2} \frac{4}{3} [1 + \sum_1^3 M_n \gamma_0^n]; \quad r_0 [\ln G_F]'' = \frac{\gamma_0^2 \langle n_G \rangle^2 z^2}{2} \frac{4}{3} [1 + \sum_1^3 N_n \gamma_0^n], \quad (19)$$

where

$$\begin{aligned} M_1 &= -(2a_1 + 0.5B + 4f_1), \\ M_2 &= a_1^2 - 2a_2 + Ba_1 - 4f_2 + 4f_1(2a_1 + 1.5B), \\ M_3 &= 2a_1a_2 - 2a_3 + 0.5B(3a_2 - B_1) - 4f_3 + 2f_2(4a_1 + 5B) \\ &\quad - 4f_1(a_1^2 - 2a_2 + 2Ba_1 + 0.75B^2), \\ N_1 &= -2a_1 - 0.5B - (1 + \frac{3}{r_0})\phi_1 + 2r_1, \\ N_2 &= a_1^2 - 2a_2 + Ba_1 - (1 + \frac{3}{r_0})\phi_2 + (1 + \frac{3}{r_0})\phi_1(2a_1 + 1.5B - 2r_1) + \\ &\quad 2r_2 + r_1(3r_1 - 4a_1 - 3B), \\ N_3 &= 2a_1a_2 - 2a_3 + 0.5B(3a_2 - B_1) - (1 + \frac{3}{r_0})\phi_3 + (1 + \frac{3}{r_0})\phi_2(2a_1 + \\ &\quad 2.5B - 2r_1) - (1 + \frac{3}{r_0})\phi_1[a_1^2 - 2a_2 + 2Ba_1 + 0.75B^2 + \\ &\quad 2r_2 + r_1(3r_1 - 4a_1 - 5B)] + 2r_3 + r_2(6r_1 - 4a_1 - 5B) + \\ &\quad r_1(4r_1^2 - 7.5Br_1 - 6a_1r_1 + 4Ba_1 + 2a_1^2 - 4a_2 + 1.5B^2). \end{aligned} \quad (20)$$

The terms with the same power of γ_0 in expressions on both sides should be equal. Therefore, one gets

$$f_1 = \frac{1}{3}[4v_1 - 2a_1 - B/2 - \frac{n_f}{4N_c}v_4 \left(1 - \frac{5}{r_0} + \frac{6}{r_0^2}\right)], \quad (21)$$

$$\phi_1 = \frac{1}{1 + 3/r_0}(f_1 + 2r_1 + 2v_7 - 2a_1 - B/2), \quad (22)$$

$$\begin{aligned} f_2 &= \frac{2}{3}f_1(4a_1 - 2v_1 + 3B) + \frac{1}{3}(a_1^2 - 2a_2 - 4v_2 - 4a_1v_1 + Ba_1 - 4Bv_1) \\ &+ \frac{n_f}{4N_c} \left\{ v_4[\phi_1 \frac{1}{r_0}(\frac{1}{3} + \frac{1}{r_0}) - \frac{2}{3}f_1 + 4\frac{r_1}{r_0}(\frac{1}{3} - \frac{1}{r_0}) + (a_1 + B)(\frac{1}{3} - \frac{5}{3r_0} + \frac{2}{r_0^2})] \right. \\ &\quad \left. - 4\frac{v_5}{r_0^2}(1 - \frac{r_0}{3}) \right\}, \end{aligned} \quad (23)$$

$$\begin{aligned} \phi_2 &= \phi_1(2a_1 + \frac{3}{2}B - 2r_1) + \frac{1}{1 + 3/r_0}(f_2 + 2r_2 + 3r_1^2 - 2a_2 - 4a_1r_1 \\ &\quad + a_1^2 - 3Br_1 + Ba_1 - 2v_7(f_1 + a_1 + B) + 4v_8 + \frac{4}{r_0}v_{10}) \end{aligned} \quad (24)$$

$$f_3 = \frac{1}{3}[d_f + \sum_{i=1}^6 d_i v_i + d_{12} v_{12} + d_{13} v_{13}], \quad (25)$$

where $d_f = M_3 + 4f_3$,

$$\phi_3 = \left(1 + \frac{3}{r_0}\right)^{-1} [f_3 + d_\phi + \sum_{i=7}^{11} d_i v_i + d_{14} v_{14}], \quad (26)$$

where $d_\phi = N_3 + (1 + \frac{3}{r_0})\phi_3$. The coefficients d_i are given in the Appendix. The numerical values of f_i , ϕ_i for different number of active flavors are shown in the Table 1.

Table 1

n_f	f_1	f_2	f_3	ϕ_1	ϕ_2	ϕ_3
3	0.364	-0.0279	0.795	0.637	-0.276	2.12
4	0.358	-0.0457	0.740	0.631	-0.286	2.04
5	0.352	-0.0629	0.689	0.625	-0.295	1.95
S	0.313	0.310	-0.120	0.313	0.310	-0.120

Herefrom it is easy to see that the asymptotical ($\gamma_0 \rightarrow 0$) values of F_2^G and F_2^F are different as has been known since long ago [10]:

$$F_{2,as}^G = \frac{4}{3}, \quad F_{2,as}^F = 1 + \frac{r_0}{3} = \frac{7}{4}. \quad (27)$$

The experimental values [3] for 41.8 GeV gluon jets, $F_2^G = 1.023 \pm 0.008 \pm 0.011$, and for 45.6 GeV uds quark jets, $F_2^F = 1.0820 \pm 0.0006 \pm 0.0046$, are much lower than the above asymptotical limits. If one accepts, however, the effective value of α_S , averaged over all the energies of partons during the jet evolution, to be 0.2, then one gets $F_2^G(NLO) \approx 1.039$ and $F_2^F(NLO) \approx 1.068$, by taking into account only the first correction proportional to γ_0 . This is quite close to the experimental results [3]. For $\alpha_S = 0.12$, one gets about 10% higher values of F_2 's. Thus we conclude that the NLO-approximation describes the widths at Z^0 -energies within 10–15% accuracy.

At lower energies the widths should be slightly smaller due to the slow increase of α_S and somewhat smaller effective values of n_f leading to larger f_1, ϕ_1 . Using the values of α_S

given in the PDG-data [11] and $n_f = 4$, we predict the energy dependence of NLO-values of second factorial moments shown by the solid curves in Fig. 1.

The additional dots at the ends of these curves demonstrate the effect due to possible change of effective values of n_f to 3 at $Q = 10$ GeV and to 5 at $Q = 90$ GeV, where Q is the total energy as in [11]. They show how small is the indefiniteness imposed by effective number of active flavours which is the only free parameter in such an approach. The curves indicate that the widths are closer to Poissonian ones at lower energies. Qualitatively, it agrees with experimental trends observed by the DELPHI collaboration [2]. The coupling strength α_S changes within the interval of Q shown here from 0.18 to 0.12. The choice of the energy scale for jets is, however, highly nontrivial (see, e.g., [2,3]). Therefore we do not plot experimental values here, just claiming the qualitative agreement within 15% accuracy.

The cut-off of the integration region at $\varepsilon = e^{-y} \approx e^{-2\pi/\beta_0\alpha_S}$ from below and at $1 - \varepsilon$ from above is not very crucial at present energies as seen from the dashed curves in Fig. 1. It diminishes the correction terms and, therefore, slightly increases the widths and flattens their energy dependence. Thus, the role of the power corrections is not very important.

Unfortunately, the higher order corrections do not improve our estimates. On the contrary, the 2NLO-term is positive and tends to violate slightly the agreement with experiment while 3NLO corrections are negative and so large that lead even to sub-Poissonian widths of distributions ($F_2 < 1$) for $\alpha_S = 0.2$. These terms are approximately equal to NLO corrections due to large values of f_3 and ϕ_3 . The inception of such large values can be traced to rather large contributions of integrals containing $\ln^2 x$, i.e. to the region of very soft gluons. Thus the cut-off at e^{-y} and $1 - e^{-y}$ becomes more important for these terms.

The 3NLO-corrections are overestimated due to the adopted Taylor series expansion with the assumption $y \gg |\ln x|$ which is invalid for soft gluons. For example, the k_t -dependence of the coupling strength is transformed so that

$$\alpha_S \propto \frac{1}{y + \ln x(1 - x)} \approx \frac{1}{y} \left(1 - \frac{\ln x(1 - x)}{y} \right), \quad (28)$$

and the second term becomes infinitely large at $x \rightarrow 0$. The cut-off at $x = e^{-y}$ leads to

a factor of 2 only. Thus the above expansion implies some special presumption about the coupling strength behavior in the nonperturbative region as well as its modification at the limits of the perturbative one. The series (12), (13) are with the sign-changing and increasing (in modulus) terms. From the values of f_i, ϕ_i in Table 1, one concludes that higher order terms are more important for the width of a quark jet compared with a gluon jet.

The slopes of the widths are especially sensitive to these higher order terms because each of them is enlarged by the factor n when differentiating γ_0^n . Thus 3NLO contribution is about 3 times larger than the NLO term in the slopes of widths. It demonstrates that any precise quantitative estimates of slopes become impossible. In particular, at present energies one cannot trust NLO estimates of these slopes as being small: $F_2^{G'}(NLO) \approx 0.04$; $F_2^{F'}(NLO) \approx 0.092$ at $\alpha_S = 0.2$. However, one can predict the asymptotical value of the ratio of slopes as

$$\frac{(F_2^G)'_{as}}{(F_2^F)'_{as}} = \frac{16f_1}{21\phi_1} \approx 0.43, \quad (29)$$

which surely coincides with their NLO ratio. It demonstrates that the second factorial moment of quark jets approaches its asymptotical value faster than for gluon jets.

Let us stress that all slopes and curvatures in pQCD are related to the running property of the QCD coupling constant since they are proportional to its derivatives which are equal to zero for a fixed coupling constant. It is interesting to note that the two-loop term in α_S proportional to β_1 contributes only to the left-hand side of Eqs (19), namely, to the coefficients M_3 and N_3 in (20), and its role is very mild there (about 1–2 %). Thus it can be accounted with high accuracy considering it only in the expressions for the expansion parameter γ_0 .

One can also easily check that all the relations of SUSY QCD (where $n_f = N_c = C_F$) are valid for all the coefficients shown above (e.g., $F_2^G = F_2^F$ etc.). The SUSY values of $f_i = \phi_i$ are shown in the lower line of the Table 1 marked by S. The asymptotical SUSY values of F_2 are equal to $4/3$.

III. THIRD MOMENTS OF THE MULTIPLICITY DISTRIBUTIONS

The system of equations (14), (15) valid up to 3NLO-terms of γ_0^3 was also applied by us to calculation of third moments of the multiplicity distributions. It was done by equating the terms proportional to z^3 on both sides of the equations and writing down the third factorial moments (defined by Eq. (9) at $q = 3$) as

$$F_3^G = h_0(1 - \sum_{i=1}^3 h_i \gamma_0^i); \quad F_3^F = g_0(1 - \sum_{i=1}^3 g_i \gamma_0^i). \quad (30)$$

Proceeding in the same way as done above, one gets $h_0 = 9/4$, $g_0 = 1 + r_0 + r_0^2/4$ and the values of the coefficients h_i, g_i in (30) shown in the Table 2. The asymptotic limit of the third moment of quark jets is about twice larger than that of gluon jets. The analytic expressions are too lengthy to be presented here.

Table 2

n_f	h_1	h_2	h_3	g_1	g_2	g_3
3	0.986	-0.342	2.49	1.61	-1.58	7.74
4	0.972	-0.380	2.36	1.60	-1.59	7.54
5	0.957	-0.417	2.25	1.59	-1.60	7.34
S	0.844	0.722	-1.09	0.844	0.722	-1.09

Comparing f_i, ϕ_i with h_i, g_i one concludes that the corrections increase for higher moments even in the NLO-approximation. Moreover, at present values of $\gamma_0 \approx 0.5$ they are rather large. Let us note the similarity in the structure of corrections for widths and third moments. They alternate in sign, and third coefficients are larger than the first ones. It is an indication on the sign-alternating asymptotic series, and Borel summation can be effective here. The increase of the coefficients originates from the terms containing the integrals of the type $\int_0^1 \ln^n x dx \propto n!$. The termination of the cascade at $\varepsilon = e^{-y}$ leading to power corrections becomes more important below Z^0 . This reminds of the situation with renormalons (see, e.g., [12]).

In SUSY QCD the asymptotical values of third moments are equal to $9/4$. The first correction given by $h_1(\text{SUSY})=g_1(\text{SUSY})=0.844$ is almost as large as for ordinary gluon jets. It is similar to the correction for second moments. However, NNLO and 3NLO-terms for moments in SUSY QCD differ drastically from those for ordinary jets both in absolute values and signs as seen from the Tables 1 and 2. It demonstrates their sensitivity to the value of r_0 which is drastically different in the two cases.

The similar procedure can be used for higher rank moments as well.

IV. GENERAL DISCUSSION AND CONCLUSIONS

The equations (1), (2) are dealing with probabilities and, therefore, are of classical nature. However, the quantum-mechanical interference has been accounted in the angular ordering effect. Nevertheless, there is no proof of their validity at all orders. The approach advocated above treats these equations as the kinetic equations in QCD for partonic cascade processes. Implicitly we have assumed that these equations describe the cascade down to extremely low energies of partons by imposing the limits 0 and 1 of integration in the shares of energy. Thus the non-perturbative region of soft partons was assumed to be described in the same manner as the perturbative cascade. Probably, one should include into the consideration only the perturbative region by the requirement that the evolution parameters under the integral in (1), (2) stay always positive i.e. by introducing the cut-off at ε and $1 - \varepsilon$ where $\varepsilon = e^{-y}$. Such a modification would lead to the power-like corrections which can be neglected asymptotically but contribute at present energies. Their role was considered for average multiplicities in [7]. They are not very important for the moments in the NLO-approximation as shown above. However, more thorough treatment is needed, especially, in view of rather large contribution of soft gluons to f_3 and ϕ_3 .

Other analytic solutions of these equations different from the perturbative one as well as nonperturbative modifications of the equations [13–15] can be looked for. Especially interesting would be to learn more about the singularities of the generating functions in the

z -plane which are known up to now for the leading order solution only.

Leaving this program for future studies, we can now compare the results obtained from perturbative solutions of the equations for average multiplicities [3-7] and for widths of the multiplicity distributions of gluon and quark jets up to 3NLO approximation of pQCD. Leading order predictions for any quantity are quite far from present experimental data. NLO corrections are always pointing in the right direction of closer agreement with experiment. In particular, the energy dependence of the gluon jet average multiplicity, of the ratio of its slopes for gluon and quark jets and the values of widths at Z^0 -energies can be fitted with rather high accuracy. However, it still fails in the ratio of average multiplicities of gluon and quark jets differing from experiment by about 30% in absolute values and even more in its energy dependence slope [7]. Here, 2NLO and 3NLO terms improve the situation so that the ratio r differs by about 10–15% only. They do not spoil good qualitative features of NLO in the energy dependence [8]. However, these corrections become very large for the widths and for the third moments as shown above, as well as for the slope and curvature in the energy dependence of average multiplicities [7].

Thus we conclude that present perturbative QCD results can describe the experimental data within 10–15% accuracy. The perturbative series breaks down, however, at different orders for different quantities. There seems to be no standard way to improve the analytic results and unanimously predict where one should truncate the expansion. Nevertheless, the general trends obtained from the perturbative approach are steadily indicating qualitative convergence of theory and experiment. Moreover, the computer results [6] show that the exact solutions of QCD equations can be even closer to experiment.

ACKNOWLEDGMENTS

This research is supported in part by the Natural Sciences and Engineering Research Council of Canada, and the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche of Québec, and also by INTAS and Russian Foundation for Basic Research.

APPENDIX:

The terms proportional to $\langle n_G \rangle^2 z^2 / 2$ in the right-hand sides of equations (14), (15) up to $O(\gamma_0^3)$ corrections can be written as

$$G_G - 1 \rightarrow F_2^G = \frac{4}{3}(1 - f_1\gamma_0 - f_2\gamma_0^2 - f_3\gamma_0^3), \quad (\text{A1})$$

$$G'_G \rightarrow 2\gamma F_2^G + F_2^{G'} = \frac{8\gamma_0}{3}[1 - (a_1 + f_1)\gamma_0 + \gamma_0^2(a_1f_1 - a_2 - f_2 + 0.5Bf_1)], \quad (\text{A2})$$

$$G''_G \rightarrow 2(2\gamma^2 + \gamma')F_2^G + 4\gamma F_2^{G'} + F_2^{G''} = \frac{16}{3}\gamma_0^2[1 - \gamma_0(2a_1 + f_1 + 0.5B)], \quad (\text{A3})$$

$$G'''_G \rightarrow 8\gamma^3 F_2^G = \frac{32}{3}\gamma_0^3, \quad (\text{A4})$$

$$\left(\frac{G_G'^2}{G_G}\right)' \rightarrow 4\gamma_0^3, \quad (\text{A5})$$

$$\begin{aligned} \left(\frac{G_F^2}{G_G}\right)' &\rightarrow \gamma_0 \frac{4}{3r_0^2}(r_0^2 - 5r_0 + 6) + \gamma_0^2 \left[\frac{8(f_1 + a_1)}{3} - \right. \\ &4 \frac{1 + r_0/3}{r_0^2}(\phi_1 + a_1 - 2r_1) - 4a_1(1 - r_0^{-1})^2 - \frac{8r_1}{r_0}(1 - r_0^{-1})] + \\ &2\gamma_0^3 r_0^{-2}[2(r_0 - 1)(2a_1r_1 + Br_1 - 2(r_2 + r_1^2)) - 2a_2(r_0 - 1)^2 + 2r_1^2 - \\ &\frac{4r_0^2}{3}(a_1f_1 - a_2 - f_2) + B((1 + r_0/3)(\phi_1 - 2r_1) - \frac{2}{3}f_1r_0^2) + \\ &2(1 + r_0/3)(2r_2 - a_2 - \phi_2 + 3r_1^2 + a_1\phi_1 - 2a_1r_1 - 2\phi_1r_1)], \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \left(\frac{G_F G_F'}{G_G}\right)' &\rightarrow 8\gamma_0^2 r_0^{-2}(1 - r_0/3) + 4\gamma_0^3 r_0^{-2}[4r_1 - \frac{r_0 r_1}{3} - \\ &(1 + r_0/3)\phi_1 - (1 - r_0/3)(4a_1 + B)], \end{aligned} \quad (\text{A7})$$

$$\left(\frac{G_F G_F''}{G_G}\right)' \rightarrow \gamma_0^3 \frac{4(9 - r_0)}{3r_0^2}, \quad (\text{A8})$$

$$\left(\frac{G_F'^2}{G_G}\right)' \rightarrow \gamma_0^3 \frac{4}{r_0^2}, \quad (\text{A9})$$

$$\left(\frac{G_G G_F'}{G_F}\right)' \rightarrow \gamma_0^2 \frac{16}{3r_0^2} + \gamma_0^3 \frac{4}{3r_0}[5r_1 - 8a_1 - 2B - (1 + 3r_0^{-1})\phi_1], \quad (\text{A10})$$

$$\left(\frac{G'_G G'_F}{G_F}\right)' \rightarrow \gamma_0^3 \frac{4}{r_0}, \quad (\text{A11})$$

$$\left(\frac{G_G G_F''}{G_F}\right)' \rightarrow \gamma_0^3 4r_0^{-2}(1 + 5r_0/3), \quad (\text{A12})$$

$$\frac{G_F^2}{G_G} - 1 \rightarrow \frac{2}{3r_0^2}(r_0^2 - 5r_0 + 6) + \gamma_0 \frac{2}{3r_0^2}(2f_1r_0^2 - (3 + r_0)\phi_1 + 4r_1(3 - r_0)), \quad (\text{A13})$$

$$\frac{G_F G'_F}{G_G} \rightarrow \gamma_0 4r_0^{-2}(1 - r_0/3), \quad (\text{A14})$$

$$\frac{G_G G'_F}{G_F} \rightarrow \gamma_0 \frac{8}{3r_0}. \quad (\text{A15})$$

The coefficients d_i in (25), (26) are

$$d_1 = 2(2a_1 f_1 - 2a_2 - 2f_2 + 3B f_1), \quad (\text{A16})$$

$$d_2 = 2(4a_1 + 2f_1 + 5B), \quad (\text{A17})$$

$$d_3 = -4, \quad (\text{A18})$$

$$\begin{aligned} d_4 = & -\frac{n_f}{2N_c r_0^2} [r_0^2 f_2 - 0.5(3 + r_0)\phi_2 - r_0^2 f_1(a_1 + 1.5B) + \\ & 0.25(3 + r_0)\phi_1(2a_1 + 3B - 4r_1) + r_0^2 a_2 - 0.5(3 + r_0)a_2 + 2r_2(3 - r_0) + \\ & r_1[r_0(2a_1 - 1.5r_1 + 3B) - 6a_1 + 9r_1 - 9B] - 1.5a_2(r_0 - 1)^2], \end{aligned} \quad (\text{A19})$$

$$d_5 = \frac{n_f}{2N_c r_0^2} [(3 + r_0)\phi_1 + 4(3 - r_0)a_1 + B(21 - 13r_0 + 2r_0^2) - r_1(12 - r_0)], \quad (\text{A20})$$

$$d_6 = -\frac{n_f(9 - r_0)}{4N_c r_0^2}, \quad (\text{A21})$$

$$d_7 = 0.5d_1, \quad (\text{A22})$$

$$d_8 = -d_2, \quad (\text{A23})$$

$$d_9 = -d_3, \quad (\text{A24})$$

$$d_{10} = -r_0^{-1}[8a_1 + (1 + 3r_0^{-1})\phi_1 + 2B(3 + 2r_0) - 5r_1], \quad (\text{A25})$$

$$d_{11} = \frac{3 + 5r_0}{2r_0^2}, \quad (\text{A26})$$

$$d_{12} = -3, \quad (\text{A27})$$

$$d_{13} = -\frac{3n_f}{4N_c r_0^2}, \quad (\text{A28})$$

$$d_{14} = 3/r_0. \quad (\text{A29})$$

The integrals v_i are as follows

$$v_1 = \int_0^1 V dx = \int_0^1 \left(1 - \frac{3}{2}x + x^2 - \frac{x^3}{2}\right) dx = \frac{11}{24}, \quad (\text{A30})$$

$$v_2 = \int_0^1 \left[\frac{\ln(1-x)}{x} - 2V \ln x(1-x) \right] dx = \frac{67 - 6\pi^2}{36}, \quad (\text{A31})$$

$$v_3 = \int_0^1 \left[\frac{\ln^2(1-x)}{x} - 2V(\ln^2 x + \ln^2(1-x)) \right] dx = 2\zeta(3) - \frac{413}{108}, \quad (\text{A32})$$

$$v_4 = \int_0^1 [x^2 + (1-x)^2] dx = \frac{2}{3}, \quad (\text{A33})$$

$$v_5 = \int_0^1 [x^2 + (1-x)^2] \ln x dx = -\frac{13}{18}, \quad (\text{A34})$$

$$v_6 = \int_0^1 [x^2 + (1-x)^2] \ln^2 x dx = \frac{89}{54}, \quad (\text{A35})$$

$$v_7 = \int_0^1 \Phi dx = \int_0^1 \left(1 - \frac{x}{2}\right) dx = \frac{3}{4}, \quad (\text{A36})$$

$$v_8 = \int_0^1 \Phi \ln x dx = -\frac{7}{8}, \quad (\text{A37})$$

$$v_9 = \int_0^1 \Phi \ln^2 x dx = \frac{15}{8}, \quad (\text{A38})$$

$$v_{10} = \int_0^1 \left[\Phi - \frac{1}{x} \right] \ln(1-x) dx = \frac{\pi^2}{6} - \frac{5}{8}, \quad (\text{A39})$$

$$v_{11} = \int_0^1 \left[\Phi - \frac{1}{x} \right] \ln^2(1-x) dx = \frac{9}{8} - 2\zeta(3), \quad (\text{A40})$$

$$v_{12} = \int_0^1 (x^{-1} - 2V) \ln x \ln(1-x) dx = \zeta(3) - 395/216 + 11\pi^2/72, \quad (\text{A41})$$

$$v_{13} = \int_0^1 [x^2 + (1-x)^2] \ln x \ln(1-x) dx = 71/54 - \pi^2/9, \quad (\text{A42})$$

$$v_{14} = \int_0^1 (1 - 0.5x - x^{-1}) \ln x \ln(1-x) dx = 1.5 - \pi^2/8 - \zeta(3). \quad (\text{A43})$$

ζ means Riemann's ζ -function.

REFERENCES

- [1] CLEO Coll., M.S. Alam et al, Phys. Rev. D 46 (1992) 4822; D 56 (1997) 17.
- [2] DELPHI Coll., P. Abreu et al, Z. Phys. C 70 (1996) 179.
- [3] OPAL Coll., K. Ackerstaff et al, Eur. Phys. J. C 1 (1998) 479.
- [4] I.M. Dremin, V.A. Nechitailo, Mod. Phys. Lett. A 9 (1994) 1471; JETP Lett. 58 (1993) 881.
- [5] I.M. Dremin, R.C. Hwa, Phys. Lett. B 324 (1994) 477.
- [6] S. Lupia, W. Ochs, Phys. Lett. B 418 (1998) 214; Nucl. Phys. Proc. Suppl. 64 (1998) 74.
- [7] A. Capella, I.M. Dremin, V.A. Nechitailo, J. Tran Thanh Van, Nucl. Phys. B (to be published).
- [8] I.M. Dremin, J.W. Gary, Phys. Lett. B (1999) (to be published); hep-ph/9905477.
- [9] I.M. Dremin, Phys. Lett. B 313 (1993) 209.
- [10] I.V. Andreev, Chromodinamika i zhestkie processy pri vysokikh energiyakh (Chromodynamics and hard processes at high energies), Nauka, Moscow, 1981.
- [11] PDG, R.M. Barnett et al, Phys. Rev. D 54 (1996) 1.
- [12] V.I. Zakharov, Talk at YKIS97, Kyoto, Dec. 1997; hep-ph/9802416; hep-ph/9811294.
- [13] I.M. Dremin, JETP Lett. 31 (1980) 201.
- [14] A.V. Leonidov, D.M. Ostrovsky, Yad. Fiz. 62 (1999) 750.
- [15] J. Ellis, K. Geiger, Nucl. Phys. A 590 (1995) 609c; Phys. Rev. D 52 (1995) 1500.

FIGURES

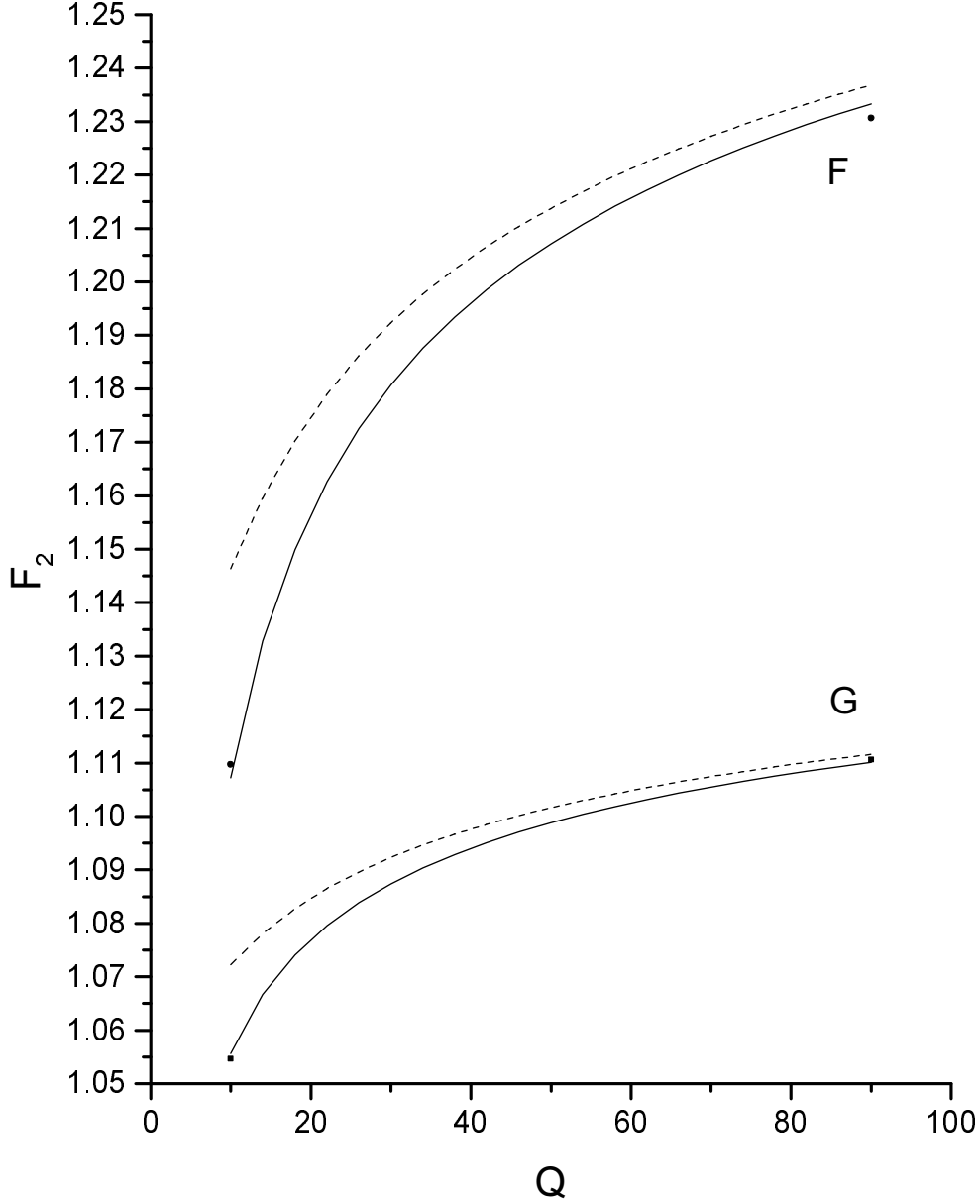


FIG. 1. The energy behaviour of the second factorial moments of quark (F) and gluon (G) jets. The limits of integration are chosen as 0 and 1 (solid lines) or ε and $1 - \varepsilon$ (dashed lines). The dots at the ends show that the curves are insensitive to variation of the effective number of flavors (see text).